Analysis of Algorithms

Midterm Exam

Code: 1988

All questions are weighted equally. All proofs must be presented formally. Assume worst case behavior and sufficiently large input unless otherwise specified. Always choose the tightest bound. If more than one answer appears correct, choose the most comprehensive answer.

Inductive proofs

Consider this statement:
\[ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \text{ for } n \geq 1. \]

Consider a proof, by strong induction and using Lusthian style, to show that the above statement is true.

1. What would we need to prove for the base case, assuming we wish to have only one base case?

   \[ \text{(A) } \sum_{k=1}^{1} k^2 = 1 \]
   \[ \text{(B) } \sum_{k=1}^{1} k^2 = 1^2 = 1 \]
   \[ \text{(C) } \sum_{k=0}^{1} k^2 = \frac{(1+1) \cdot (2+1)}{6} \]
   \[ \text{(D) } \sum_{k=0}^{1} k^2 = \frac{(1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6} \]
   \[ \text{(E) } \sum_{k=2}^{2} k^2 = 2^2 = 4 \]
   \[ \text{(F) } \sum_{k=2}^{2} k^2 = \frac{(2 \cdot (2 + 1) \cdot (2 \cdot 2 + 1)}{6} \]
   \[ \text{(G) } \sum_{k=1}^{1} k^2 = \frac{(1 \cdot (1+1) \cdot (2 \cdot 1 + 1}{6} \]

2. What would the inductive hypothesis be, assuming \( b \) is the largest base case value?

   \[ \text{(A) } \text{Assume true for } b \leq i \leq n \]
   \[ \text{(B) } \text{Assume true for } b \leq n \]
   \[ \text{(C) } \text{Assume true for } b \]
   \[ \text{(D) } \text{Assume true for } b \leq i < n \]
   \[ \text{(E) } \text{Assume true for } b < i \leq n \]
   \[ \text{(F) } \text{Assume true for } b < i < n \]
   \[ \text{(G) } \text{Assume true for } n \]
   \[ \text{(H) } \text{Assume true for } b < n \]

3. Assume \( A \) is equal \( n+1 \) while \( B \) is equal to \( n-1 \). If the first line of the inductive case sets up the inductive step on the next line, then the first line should be:

   \[ \text{(A) } \sum_{k=1}^{n} k^2 = n^2 + \frac{B(B+1)(2B+1)}{6} \]
   \[ \text{(B) } \sum_{k=1}^{A} k^2 = 1^2 + \sum_{k=2}^{A} k^2 \]
   \[ \text{(C) } \sum_{k=1}^{A} k^2 = A^2 + \frac{n(n+1)(2n+1)}{6} \]
   \[ \text{(D) } \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} + \frac{B(B+1)(2B+1)}{6} \]
   \[ \text{(E) } \sum_{k=1}^{n} k^2 = 1^2 + \sum_{k=2}^{n} k^2 \]
   \[ \text{(F) } \sum_{k=1}^{A} k^2 = A^2 + \sum_{k=1}^{n} k^2 \]
   \[ \text{(G) } \sum_{k=1}^{A} k^2 = \frac{A(A+1)(2A+1)}{6} + \frac{n(n+1)(2n+1)}{6} \]
   \[ \text{(H) } \sum_{k=1}^{n} k^2 = n^2 + \sum_{k=1}^{B} k^2 \]

Master recurrence theorem (MRT)

The master recurrence theorem refers to the one described in Cormen, et al.,
4. In terms of the master recurrence theorem, where does the equation \( T(n) = 6T\left(\frac{n}{2}\right) + \frac{n^3}{\log \log n} \) fall? The log (base 6) of 2 is ~0.387. The log (base 2) of 6 is ~2.585.

(A) case 2  
(B) the equation does not fit the MRT  
(C) case 3

5. In terms of the master recurrence theorem, where does the equation \( T(n) = 4T\left(\frac{n}{3}\right) + n \log \log n \) fall? The log (base 4) of 3 is ~ 0.792. The log (base 3) of 4 is ~ 1.262.

(A) case 2  
(B) between case 1 and case 2  
(C) the equation does not fit the MRT

6. In terms of the master recurrence theorem, where does the equation \( T(n) = T\left(\frac{n}{10}\right) + 1 \) fall?

(A) case 3  
(B) the equation does not fit the MRT  
(C) case 1

7. In terms of the master recurrence theorem, where does the equation \( T(n) = 5T\left(\frac{n}{5}\right) + n \log \log n \) fall?

(A) the equation does not fit the MRT  
(B) between case 2 and case 3  
(C) case 1

8. Consider using the master recurrence theorem to solve the equation \( T(n) = 3T\left(\frac{n}{\sqrt{2}}\right) + 1 \). What is the largest listed value of \( \epsilon \) that can be used in the solution? The log (base \( \sqrt{2} \)) of 3 is ~ 3.170. The log (base 3) of \( \sqrt{2} \) is ~ 0.315.

(A) no legal value is listed  
(B) 0.3  
(C) \( \epsilon \) is not used in the solution

9. Consider using the master recurrence theorem to solve the equation \( T(n) = 3T\left(\frac{n}{\log_2 n}\right) + n \). What is the largest listed value of \( \epsilon \) that can be used in the solution? The log (base 2) of 3 is ~ 1.585. The log (base 3) of 2 is ~ 0.631.

(A) no legal value is listed  
(B) 0.6  
(C) 0.15

10. Consider using the master recurrence theorem and the regularity condition for case 3 to solve the equation \( T(n) = 8T\left(\frac{n}{2}\right) + n^3 \). What is the smallest legal value of the constant \( c \) listed for the solution? The log (base 8) of 2 is ~ 0.333. The log (base 2) of 8 is 3.

(A) 0.3  
(B) 1.1  
(C) 0.8

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**Binary search trees and self-balancing search trees**

References to binary search trees, red-black trees, and AVL trees refer to typical implementations, unless otherwise specified. For red-black trees, null pointers are not considered nodes but do have a color (black). In a red-black tree, all nodes being inserted start out red.
11. Consider a right-rotation of a node $n$ upwards in a BST. The former sibling of $n$, assuming it exists:

(A) becomes the right child of $n$
(B) becomes a grandchild of $n$
(C) remains the sibling of $n$
(D) becomes the left child of $n$
(E) becomes the niece or nephew of $n$
(F) none of the other answers are correct

12. Suppose you allowed, at most, three red nodes in a row on any path from an interior node to a leaf in a red-black tree (as opposed to a standard red-black tree where you can have, at most, one red node in a row). Of the answers listed, what is the tightest upper bound on the ratio of the longest path from the root to a leaf and the shortest path from the root to a leaf as the length of the paths tend toward infinity. Also, will searches and insertions would still take $O(\log n)$ time?

(A) 4, no
(B) 4, yes
(C) infinity, no
(D) none of the other answers are correct

13. The number of node rotations that occur after an insertion into a red-black tree is:

(A) $\Theta(1)$
(B) $\Theta(\log n)$
(C) $\Theta(n \log n)$
(D) $\Theta(n)$

14. Suppose you made $n$ insertions in a row, $n > 1$, into an empty red-black tree. For some orderings of the $n$ values, can an all-black tree result? Let the red uncle be case 1, the black uncle case 2, and $x$ be the newly inserted red node.

(A) none of the other answers are correct
(B) no, because cases 1 and 2 leave behind a red node
(C) no, because the inserted node always remains red
(D) yes, because the root is colored black at the end
(E) yes, because case 1 can color $x$ black (eventually)
(F) yes, because case 2 can color $x$ black (eventually)

15. Inserting the integers 1, 2, 3, and 4 into an empty red-black tree, in the order given, yields how many recolorings in total? Don’t forget to color the root black, if need be. Do not count the initial coloring of the inserted node.

(A) none of the other answers are correct
(B) 4
(C) 8
(D) 0

16. Consider inserting the numbers 1, 2, and 3 into an empty red-black tree in all possible ways (e.g. 3, 2, 1 and 2, 1, 3). How many of these orderings cause a double rotation?

(A) 6
(B) 0
(C) 2
(D) 1

17. Consider a node $x$ with two children $a$ and $b$, in a red-black tree. If $a$ is a leaf, what is the most number of descendants $b$ can have?

(A) 3
(B) 2
(C) 7
(D) 4

(E) none of the other answers are correct
(F) 1
(G) 5
(H) 6
18. Consider an AVL tree having a root whose left side has 4 nodes. What is the most number of nodes that can appear on the right side?

(A) none of the other answers are correct
(B) 7
(C) 16
(D) 15
(E) 8
(F) 12
(G) 11
(H) 9

19. T or F: Consider a node \( x \) with a single child in an AVL tree. It is possible for \( x \)'s child to be a parent as well.

20. Consider a node with two children in an AVL tree. If one of the children is a leaf, what is the most and fewest number of descendants the other child can have, respectively?

(A) 2 and 2
(B) 2 and 1
(C) 3 and 1
(D) 1 and 0
(E) 1 and 1
(F) none of the other answers are correct
(G) 3 and 0
(H) 2 and 0

21. Consider inserting the following numbers, in the order given, into an empty AVL tree and then performing a level-order traversal of the tree in which the balance factors are reported:

1 2 3 6 5 4

Which of the following sequences of balance factors would result from that level-order traversal?

(A) 0 1 0 -1 0 0
(B) 0 1 -1 0 0 0
(C) -1 -1 0 0 0 0
(D) 1 0 -1 1 0 0
(E) none of the other answers are correct
(F) 0 1 0 0 0 0
(G) 0 -1 0 0 0 0
(H) -1 1 0 0 0 0

22. Consider inserting the following numbers, in the order given, into an empty AVL tree:

1 2 3 6 5 4

Which insertions cause a single rotation and which cause a double rotation?

(A) there are no rotations
(B) single: 3, 6, 5
(C) single: 3, double: 4
(D) single: 3, double: 6
(E) double: 3, 5, 4
(F) none of the other answers are correct
(G) single: 3, double: 6, 4
(H) single: 3, double: 5, 4

Binomial heaps

Assume min-heaps unless otherwise directed. For binomial heaps, assume that root lists (and child lists) are ordered left-to-right from low degree to high degree. Also assume when a child list is merged with a root list, the child list is placed to the left of the root list (i.e. the original root list is scanned after the child list during the consolidation phase).

23. Consider inserting the following values, in the order given:

9 2 8 5 6 1 4 7 3 0

into an empty binomial heap. After deleting the value 8, the parent of 9 is:

(A) 0
(B) 3
(C) 2
(D) 4
(E) 1
(F) 5
(G) 6
(H) none of the other answers are correct
24. Consider inserting the consecutive integers from 0 to 12, inclusive and in increasing order, into an empty binomial heap. After three \textit{extractMin} operations, how many subtrees are left in the heap?

(A) none of the other answers are correct  
(B) 3  
(C) 1  
(D) 4  
(E) 5  
(F) 2

25. Consider inserting the consecutive integers from 0 to 14, inclusive and not necessarily in order, into an empty binomial heap. After all the insertions, what is the largest possible root value of the largest heap in the root list?

(A) 7  
(B) 10  
(C) 9  
(D) 8  
(E) 12  
(F) 6  
(G) none of the other answers are correct  
(H) 11

**Fibonacci heaps**

Assume min-heaps unless otherwise directed. Assume that root lists (and child lists) are ordered left-to-right from low degree to high degree. Also assume when a child list is merged with a root list, the child list is placed to the left of the root list (i.e. the original root list would be scanned after the child list if consolidation were to happen). When inserting, assume the newly inserted node is placed at the front of the root list.

26. Consider inserting the numbers 0 through 11, inclusive and in any order, into an empty Fibonacci heap. What is the largest value that could possibly be a root?

(A) 3  
(B) 5  
(C) 11  
(D) 4  
(E) 7  
(F) 8  
(G) none of the other answers are correct  
(H) 6

27. Consider inserting the following values, in the order given:

\[3 2 9 5 6 4 1 0\]

into an empty Fibonacci heap. After an \textit{extractMin} operation, the value 6 is found in a subheap whose root has value:

(A) 6  
(B) 0  
(C) 4  
(D) 3  
(E) 2  
(F) 5  
(G) none of the other answers are correct  
(H) none of the other answers are correct

28. Consider inserting the consecutive integers from 0 to 12, inclusive and in increasing order, into an empty Fibonacci heap. After six \textit{extractMin} operations, the value 12 can be found in the subheap whose root has value:

(A) 7  
(B) none of the other answers are correct  
(C) 9  
(D) 5  
(E) 6  
(F) 8

29. Consider this set of operations: 12 inserts and one extraction of the minimum (in any order). What is the fewest / most number of subheaps found after the set is performed on an initially empty Fibonacci heap? Note: you need at least one insert before performing an extraction.

(A) 3 / 9  
(B) 3 / 11  
(C) 3 / 12  
(D) 2 / 11  
(E) 2 / 9  
(F) none of the other answers are correct  
(G) 3 / 10  
(H) 2 / 10
30. What is the most number of marked nodes that can exist in a path from a degree 6 subheap root to a leaf? Hint: extrapolate from smaller subheaps.

(A) 3  
(B) 2  
(C) 16  
(D) none of the other answers are correct

The following questions assume an up-tree implementation of a disjoint set. Assume each value has a pointer to its parent with the root of the tree serving as the representative of the set. When performing a union between two trees of equal rank, assume the larger valued root becomes the parent of the lesser valued root. Remember, the union operation calls findSet to find the representatives of its two arguments.

31. In the worst case, the find-set operation (with path compression but no union by rank) takes what kind of time?

(A) logarithmic  
(B) quadratic  
(C) constant  
(D) linear  
(E) none of the other answers are correct  
(F) log linear

32. In the worst case, the find-set operation (with no path compression but union by rank) takes what kind of time?

(A) logarithmic  
(B) log linear  
(C) linear  
(D) none of the other answers are correct  
(E) constant  
(F) quadratic

33. In the worst case, the union operation (with no path compression and no union by rank) takes what kind of time when two representatives are passed into union?

(A) linear  
(B) constant  
(C) logarithmic  
(D) quadratic  
(E) log linear  
(F) none of the other answers are correct

For the following set of questions, consider the following sequence of operations:

```
for each i in 0..9 do make-set(i) //10 sets initially
union(0,1);
union(1,2);
union(3,4);
union(5,6);
union(3,5);
union(2,4);
union(7,9);
union(7,8);
find-set(2);
find-set(9);
union(3,8);
```

assuming union by rank and path compression. When unioning two sets having the same rank, assume the root with the larger value becomes the root of the resulting set.

34. Immediately after the union(7,8) operation, how many nodes are more than one step away from a root?

(A) none of the other answers are correct  
(B) 1  
(C) 4  
(D) 3  
(E) 2  
(F) 5
35. Immediately after the \textbf{find-set(2)} operation, how many children are one step away from a root and how many children are more than one step away, respectively?

(A) 5 and 4  
(B) 2 and 6  
(C) 4 and 4  
(D) 6 and 2  
(E) 5 and 3  
(F) 3 and 5  
(G) none of the other answers are correct  
(H) 7 and 1

36. Immediately after the \textbf{union(3,8)} operation, how many children does 4’s representative have?

(A) 2  
(B) 4  
(C) none of the other answers are correct  
(D) 6  
(E) 5  
(F) 3

Graphs
The length of a walk, tour, path, or cycle is the number of edges on that walk, tour, path, or cycle.

37. If a vertex in an undirected, simple, graph with at least ten vertices is on a cycle, the degree of that vertex cannot be less than:

(A) none of the other answers are correct  
(B) 2  
(C) 1  
(D) 3  
(E) 9  
(F) 4  
(G) 0

38. What is the length of the longest possible path in a simple graph?

(A) the number of vertices plus 1  
(B) the number of edges minus 1  
(C) the number of vertices  
(D) the number of edges  
(E) the number of edges plus 1  
(F) the number of vertices minus 1

39. Consider these statements:

\begin{enumerate}
\item every simple, connected graph has a Hamiltonian path
\item every simple, complete graph has a Hamiltonian path
\end{enumerate}

Which are true?

(A) neither  
(B) both  
(C) only the second  
(D) only the first

40. In a simple, directed graph, the most number of edges in a graph with \( V \) vertices is:

(A) \( V^2 \)  
(B) \( \frac{(V^2 - V)}{2} \)  
(C) \( V^2 - 1 \)  
(D) \( V^2 - V \)  
(E) none of the other answers are correct  
(F) \( 2V \)  
(G) \( \frac{V^2}{2} \)  
(H) \( \frac{(V^2 - 1)}{2} \)